GeoGebra affords a variety of representational resources and computational utilities that have the potential to engage prospective mathematics teachers in insightful investigations of fundamental ideas of mathematics. Given that many prospective mathematics teachers have a fragmented view of mathematics, GeoGebra, as an integrated system of algebra, geometry, and calculus, stands as a mathematically enriched environment where prospective teachers could explore and/or construct mathematical objects and further develop a connected view of mathematics. In this article, we share our experience in using GeoGebra with prospective middle and secondary mathematics teachers, highlighting the need for reconceptualization and remediation for both the instructor and the prospective teachers.

Initial Experience with GeoGebra

We initially introduced GeoGebra to our middle and secondary prospective teachers in Spring 2007, when we were teaching a methods course on the use of technology in mathematics education at a large public university in southeast US. The primary objective of the course is to engage prospective teachers in exploring and reflecting on the multiple uses of instructional technologies in the middle and secondary grades in order to foster their awareness and subsequent development of technological pedagogical content knowledge (TPCK) for mathematics teaching [5, 6]. The entire course was implemented in a computer lab where all the computers were preloaded with the Geometer’s Sketchpad™ (GSP) at that time. Of their own choices, the instructor and the prospective teachers quickly switched to the open source GeoGebra. While both the instructor and the prospective teachers were learning about the features of GeoGebra, the class tackled a variety of “bite-size” problems [7], including triangle area, curve-fitting, and a few geometric construction problems. The reality was that our prospective teachers did not learn their mathematics in a technology-supported environment at schools, and thus they had no prior experience in connecting the different fields of mathematics such as arithmetic, algebra, geometry, and calculus with technology. Although many of them could recognize the significance of technology in the teaching and learning of mathematics, they tended to believe that mathematics should be first taught with traditional methods and then be explored using technology, revealing challenges associated with all dimensions of TPCK.

Moving toward a model-centred instructional framework

As we explored the representational infrastructure of GeoGebra and its new affordances, we began to plan, in the following semesters, for the use of GeoGebra in a holistic and
integrated perspective using the model-centered approach [4, 8, 9]. We focused on supporting prospective mathematics teachers’ conceptual changes in both content and pedagogy. With its construction tools and multiple representations, GeoGebra provides an accessible platform to model and further simulate a variety of mathematical ideas, allowing for both expressive and exploratory ways of model-based learning [2]. Our subsequent work with GeoGebra was informed by the theory of model-centred learning and instruction [8] and guided by the instructional design framework of model-facilitated learning [4]. Accordingly, we envisioned that prospective teachers would be able to reconceptualize big ideas of mathematics in meaningful contexts, where model building served as a means to bridge the different levels of mathematical understanding and, meanwhile, control the problem complexity. Prospective teachers, in a model-centered learning perspective, would have the opportunity to construct situation-bound mental models to make sense of the problems, using GeoGebra utilities as cognitive tools to facilitate their model building and investigations. GeoGebra-based models were further used to support communications within the learning community.

An instructional sequence on quadratic relations

In what follows, we present an instructional sequence we developed around the idea of quadratic relations in a semester-long study. Since almost all of our prospective teachers had knowledge of the quadratic function and the parabola, we decided to provide various conceptually engaging activities [3] for them to develop a holistic view of quadratic relations. The instructional sequence starts with a paper folding activity, followed by GeoGebra-based modeling, curve-fitting, reasoning, construction, and higher-level analyses. Along the geometric dimension, we also included the parabola construction from its definition.

The paper folding activity

This is a classic example of parabola construction (Fig 1). Given a piece of paper, the prospective teachers were guided to fold out an envelope of creases that give rise to the shape of a parabola. This paper folding process prepared them for an in-depth exploration of quadratic functions in the GeoGebra environment. In a traditional setting, it is difficult to take the activity to a higher level because learners sometimes cannot gather enough evidence to determine what they see is a parabola.

Modelling and simulating the paper folding activity

To model the paper folding activity, it is important that the prospective teachers identify the essential components of the activity. A model is not a mirror image of its physical counterpart but rather a higher-level representation of one’s interpretation of a certain phenomenon in the world [3]. In an attempt to help the prospective teachers determine how the paper folding process could be transferred onto the GeoGebra drawing pad, we held a whole-class discussion which led to the understanding that what they needed was just a line (for the edge of the paper) and a point close to the line. The following questions in the next step of the process were challenging to the prospective teachers: “What does it do when you pick up a point on the line and fold it toward the point above the line? What do you know about the crease?” When they realized that the crease was just the bisector of the segment connecting the two points (Fig 2), they built a GeoGebra model to simulate the paper-folding process (Fig 3). We note that most of the prospective teachers tended to struggle with the concept of dependency in the GeoGebra environment. From a cognitive perspective, dependency resolution seems to be an essential part of model building and one’s growing mathematical understanding of the structural relations in a problem scenario.

Is that really a parabola?

The paper folding model in GeoGebra allowed further questions to be addressed about the curve produced by the creases. We asked the following questions to scaffold the prospective teachers’ further inquiry into the problem. “Is it really a parabola?” “Is there really a curve there?” “If there is a curve there, how are the lines related to that curve?” These questions led the prospective teachers to comment that all
the lines were tangent to the parabola curve. Since most of our prospective teachers had learned about calculus, we decided to plot the slope of the creases when the arbitrary point moved along the edge of the paper. Using the slope feature of GeoGebra, the prospective teachers made a graph of the slopes, which turned out to be a straight line (Fig 4). In the subsequent discussions, they came to be convinced that the curve formed within the lines was indeed a parabola (a quadratic function as the integral of a linear function) with a gentle reference to calculus.

Curve-fitting activities
Building on prospective teachers’ prior knowledge of functions, curve-fitting activities afford opportunities for them to investigate a function’s general behavior and the significance of function coefficients. In the case of the parabola, most of our prospective teachers knew that it was related to the standard form of the quadratic function: \( f(x) = ax^2 + bx + c \). A quadratic function, for example, could be matched to the parabolic curve in the previous paper folding activity. The water fountain problem was another case to achieve similar objectives. Using GeoGebra sliders for the coefficients \( a, b, \) and \( c \), our prospective teachers explored how changing coefficients affected the graph of a quadratic function.

The focus-directrix definition
A parabola is also defined as a collection of points in the plane that are equidistant from a point (the focus, \( F \)) and a line (the directrix). While this definition describes the properties of a parabola, it does not provide any direct information on its construction. In this regard, thinking backwards may be a good strategy to establish the relationship between the directrix, points on the parabola, and the focus. This leads to the fact that for each point \( D \) on the directrix, there corresponds a point \( P \) on the parabola that falls at the intersection between the bisector of \( DF \) and the line that is perpendicular to the directrix at point \( D \) (Fig 5). Taking dependency into account, the construction should start with an arbitrary point on the directrix. Such analysis is an essential part of problem solving in geometric constructions, which require the prospective teachers to make use of their geometric knowledge. The validity of such analysis can be tested and confirmed by subsequent constructions (Fig 6). While GeoGebra is of little help in such analysis, it serves as a tool for meditating the technical process of construction and the cognitive processes of mathematical reasoning as the prospective teachers internalize such models.

Interestingly, the focus-directrix construction will reveal a family of lines that are similar to those in the paper folding activity at the beginning of the instructional sequence. A series of questions could be asked about the relationship between the two activities. If, in the paper folding activity, each crease contributes one point to the parabola, where is that point located? Is the lower edge of the paper the directrix of the parabola? Indeed, as the prospective teachers discuss the activities, they would find that the two could well be combined in that each crease in the paper folding activity contains one point that is equidistant to the lower edge of the paper and the point above. In our teaching experience, we found that our prospective teachers needed considerable scaffolding in developing a constructive understanding of the definition and resolving the dependencies.

Concluding remarks
GeoGebra provides a variety of representations and technical utilities that can be used to engage prospective mathematics teachers to relearn, reorganize, and remediate their knowledge of mathematics [1]. While GeoGebra can be used to support the teaching and learning a variety of mathematical ideas, it is pedagogically advisable to take advantage of its multiple dynamic representations in designing holistic model-centered instructional sequences to engage prospective mathematics teachers in developing
a TPCK that is crucial for their future classroom teaching. The parabola sequence is one of our first attempts to use GeoGebra as a tool to reconceptualize the teaching of fundamental ideas in school mathematics. We recognize that the same ideas may be investigated in alternative ways and the sequence was not intended to be exhaustive in nature but rather to provide a specific case as an illustration of GeoGebra-based remediation. In a recent effort, we started to use concept mapping as a tool for prospective teachers to externalize their emergent mathematical reasoning, understand the nature of their own mathematical learning, and manage the complexity of learning tasks. A tentative concept map for the quadratic relation is shown in Fig 7. To conclude, we call for mathematics educators to consider the mutually constitutive relationships between dynamic multiple representations and specific learning tasks [2], seeking a synergy between dynamic representations and mathematical concepts and that between specific mathematical ideas and general ways of mathematical thinking [1].

References


