Introduction

Many years ago, I went up to Cambridge with the words of the Head of Maths ringing in my ears: ‘At Cambridge you will be able to study maths for the sake of studying maths!’ The implied contrast was with A-level preparation which consisted solely of working through old exam papers.

Unfortunately, he was wrong, because our university course consisted of the same: lectures plus working through old papers. It was content, content and more content, full stop. I knew that down the road English students were enjoying seminars with distinguished scholars, discussing, arguing, speculating... but that was not for us. Why was topic X important? Where did Y come from? What was the history of Z? Why were A, B and C connected? What was the motivation behind method P and why was it superior to Q? Silence. Such questions were not answered, they were not even asked.

By chance, my maths tutor for a couple of terms was the great Frank Adams but we still went through old exam papers. I did tentatively suggest to him that I would like to solve a problem. His response was to suggest that I study Dirichlet’s proof that every arithmetic progression (AP), excluding the trivial exceptions, contains an infinity of primes and then to try to reproduce it. Well, I’m all in favour of reading the masters, but that wasn’t what I meant.

It has always been my hope that before I am dead, mathematics teaching at Oxbridge will have moved on, but reading ‘Promoting Mathematics’ by Ronnie Brown, (to which this article is a response), I am not confident. He quotes Dr Brian Stewart of Oxford:

“My own experience is that mathematicians do almost always discuss [education] in terms of ‘content’; how much can we pack in. This is very odd, because it doesn’t match how we speak to each other about how we learn and develop new ideas / understandings.” [3]

Brown goes on to write:

“An emphasis on content and assessment may ... give students the impression that the highest achievement for a mathematician is to write many neat answers to examination questions.”

Indeed. In my experience of teaching mathematics in primary and secondary schools, pupils all too easily learn the same lesson, that their highest achievement is to answer exercise questions correctly and then do the same in official examinations. I disagree. To educate pupils mathematically, rather than merely train them, you cannot safely or successfully ignore background, setting, context or perspective. Here are seven
perspectives from which, I believe, teaching (and learning) mathematics can be profitably viewed, whatever the level of the student.

What's the point of matrices, Sir?

I was once asked this question by a 13-year old School Mathematics Project (SMP) pupil. Dissatisfied with my answer, I did a quick survey of the other members of staff and discovered that they had no good responses either, which was disturbing because this was the height of the Modern Maths Movement and matrices were everywhere, promoted by the Bourbaki insistence that linear algebra was far more important for pupils than Euclidean geometry. Textbooks typically say little or nothing about the reasons for studying a topic, as if they were obvious or unimportant, but plausibly they are neither. University curricula include dozens of topics and themes, but leave out dozens more. Why? Why not put students in the picture, give them some history, explain the motivating problems that led to the creation of the field, sketch out links to related themes?

It’s true that not all students will appreciate your solicitude, but many will. Boys and girls differ in their typical learning styles. One distinction is between learners who are ‘serialist’ and those who are ‘holist’ [1].

“Serialists’ prefer to accumulate new knowledge in sequenced chunks, and with set procedures and methods for problem solving. ‘Holists’ tend to assimilate new knowledge within existing understanding, looking for connections and similarities within the overall ‘big picture’ context of their learning. The studies report that a disproportionate number of boys exhibit holistic tendencies, and that it has been established that the level of uncertainty at which individuals are happy to work is a distinguishing characteristic between serialists and holists.”

What this says about girls is uncertain: they could simply be clinging to a serial style of teaching out of anxiety, but the message from boys is clear: many would benefit from a style which they do not get, at school or university.

Different perspectives on mathematics

More than a decade ago, Ronnie Brown argued that:

“If it is unclear as to what is mathematics, and what are its main achievements, what constitutes performance in it, then what is the hope of teaching it in a clear way, of [understanding] how it should be taught more effectively>?” [2]

Professional philosophy of mathematics is highly technical but the philosophy of maths education is not, and influential figures from Polya to Lakatos to Hersh & Davis have all had their say over the last 50 years or so [6]. Students do not need to study such figures but lecturers plausibly should, not least in view of the explosion of interest in experimental mathematics and (claimed) changing attitudes to proof: the students of today may have been using graphics calculators and computers from the start of their secondary schooling, or even earlier.

Is mathematics - should it be - about problem solving? About applying mathematics to the world? About creating new concepts? About exploring mathematical landscapes?

Ironically, the problem solving approach, which might seem perfectly suited to mathematics teaching has made little inroads but has been adopted successfully in very different subjects, such as medicine where it has been very successful and students and teachers have benefited.

Aesthetics: from beauty to ‘cool’

The greatest mathematicians have all witnessed to the beauty of mathematics, which is also a theme in many popular books - but not in the classroom or the lecture theatre. Why not?
I recently had the chance to judge the aesthetic sensibility of a talented 13-year-old pupil of mine by asking him to rate sets of problems or theorems from 1 to 10 by their ‘attractiveness’, and his aesthetic judgements often matched mine fairly closely. For example, given the problem of taking two maps of identical regions but on different scales, placing the smaller map on top of the larger and then asking for two coincident points representing the same location, he gave it 9/10 before tackling it (successfully) or indeed having any idea what it required. However, he did describe such problems as ‘cool’ rather than beautiful!

Secondary pupils, or university students, however, are rarely expected to make any such judgments, let alone to act on them. It is true that mathematicians vary considerably in the judgements they make [4, 5] but that is a reason for discussing such differences with students, not ignoring them.

Behaving like a mathematician

Many years ago, when teaching secondary pupils, it was my custom to put large numbers of problems - simple ones, inevitably, because the pupils were weak - onto cards, laid them on a table top, and instructed pupils that:

1. they were to choose a problem that appealed to them and tackle it; and,

2. if they chose a problem and then changed their minds about it, they were allowed to exchange it for another one. Indeed, they ought to do so, because that’s what professional mathematicians did.

I actually spent quite a lot of time explaining (in suitably simple terms) what professional mathematicians would do, on the grounds that whereas youngsters, girls as well as boys, have a good idea what happens on a football pitch or a tennis court, and they know something about rock climbing and mountaineering, all being highly visual and visible activities, they had no idea at all what mathematicians did, maths being almost totally invisible. Thus game players faced with a choice of opponents, and rock climbers faced with a choice of routes from easy to very severe, make choices that are neither so easy as to offer no challenge, nor so difficult that you are bound to fail miserably: they choose goals that are within their reach, just, and the same should be true, I explained, of mathematical problems, and they understood this point perfectly well.

They also grasped many other simple points about ‘how mathematicians behave’. These amounted, in most cases, to no more than simple heuristics, but by expressing them in terms of ‘what a mathematician would do’, the impact seemed to be far greater. Thus to pupils with a disorganised mass of information, I might suggest that ‘a mathematician would organise the information into a table ...’ (not that the pupil should put the information into a table) and this small difference seemed to make a big difference to the pupil’s reaction. I concluded that pupils benefited by ‘seeing’ themselves as young mathematicians.

“Mathematics is an extraordinarily rich and imaginative activity, a science and an art rolled into one, but this richness is replaced for pupils and students at various levels of education by a dull and unimaginative dryness, created by a system of teaching and assessment in pursuit of content, content and yet more content.”

Levels of learning: pure creativity to rote algorithms

There are many analogies between chess and mathematics: one is that there are different levels of learning in both, and that it can be useful to learn something - the best move in an opening sequence, for example - long before you are capable of understanding deeply why it is the best move. Indeed, it could be argued that most chess players will never ever understand, at the ‘deepest’ level, why after 1: e4, e5 2:Nf3, Nc6 3: Bb5, the commonest and recommended move is 3 ... a6.

In other words, most pupils start learning ‘by rote’ that 3 ... a6 is a good move, and then their understanding develops slowly as a result of experience, which suggests von Neumann’s response to a student who claimed not to have understood a point in his lecture, ‘You don’t understand mathematics, you just get used to it.’ Nonsense in a sense but with a crucial kernel of truth!

So I share Ronnie Brown’s query: ‘Is there a danger that the baby of practice has been thrown out with the bath water of rote learning’ [3].

However, I think that a large part of the explanation is that teachers generally see nothing occupying the pedagogical space between pure rote learning and children understanding deeply. Since pure rote learning is well-known to produce really bad results, they reject rote learning and focus on ‘deep’ understanding, ignoring the intermediate arena in which pupils do not understand deeply but can explore and develop slowly but surely what understanding they have.

There is another analogy here with chess in which there is a vast region of possibilities between playing by rote and being extremely creative. This is the arena of methods or as a chess player might say, tactics and strategies.

Particular areas of maths as they become more developed and better understood, generate methods that must be applied in a game-like manner - I am thinking for example
of that old textbook, *Methods of Mathematical Physics* by Jeffreys and Jeffreys. Such methods cannot be applied by strict rules, they are not algorithms, but they can be used with an appropriate understanding of tactics and strategy.

Mathematicians, pure and applied, develop a feeling for these game-like techniques and the multitude of (often both powerful and elegant) tactics and strategies involved. So I agree with Ronnie Brown also that:

“One aim of mathematics is to set up machinery of which you do not need to know or test all the parts before you use it, just as you can drive a car without knowing the workings of the internal combustion engine. This is the function of lemmas and theorems.” [3]

but with the added qualification that it is also the function of game-like mathematical methods.

**Creating new concepts**

Ronnie Brown has also emphasised the importance of students creating concepts, and understanding that process. Even this possibility exists for younger pupils, though of course it is harder and opportunities occur less frequently. An example is,

Given a ball which shoots out of the corner of a rectangular billiard table with integral sides and bounces at 45 degrees off the sides, what will happen?

One solution, popular with teachers, uses reflection to find a ‘clever’ answer but a pupil years ago argued that, supposing the table is 5 by 7, the ball will traverse the length of the table a whole number of times before disappearing into a corner, so the total distance travelled will be one of 7, 14, 21, 28 ... But it will also traverse the width of the table, ditto, so the answer is also in the sequence, 5, 10, 15, 20 ... This pupil had ‘very nearly’ invented the concept of the lowest common multiple, in solving a problem.

That boy was 12 or 13 years old. Should not much older students be able, albeit hesitantly and at first incoherently, to find their way to important concepts by exploring and solving problems?

**Conclusion**

Mathematics is an extraordinarily rich and imaginative activity, a science and an art rolled into one, but this richness is replaced for pupils and students at various levels of education by a dull and unimaginative dryness, created by a system of teaching and assessment in pursuit of content, content and yet more content. Millions of members of the public will happily (and correctly, up to a point) claim that they appreciate art and architecture, films, painting and literature, but how many will claim to appreciate mathematics?

Numbers of pupils taking A-level Mathematics are falling year by year and we are told that we need more maths graduates. Maybe the time is coming when the belief that mathematics can and should be taught as content-without-appreciation will be seen as totally unacceptable and the necessity of taking into account the aspects of maths sketched here will be taken for granted. Maths students could then be described as educated rather than merely trained, and mathematicians when asked at a party, ‘So, what do you do?’ would be able to do more than reply, ‘Oh, I’m a mathematician’ and quickly move on to another topic of conversation.

**References**


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