There is substantial anecdotal evidence to suggest that students entering many mathematical sciences courses in HE are less well prepared than previously. At Keele University, we have been using the same diagnostic test to help students in their transition from School to University; the results over the last 15 years show substantial variation but no statistically significant fall in the average mark scored. In a small pilot study we set out to compare students’ mechanical skills with their conceptual understanding in order to understand this apparent dichotomy.

1. Introduction

There has been considerable discussion on the mathematical ability of students entering UK Mathematics departments. The evidence presented by A-level results is somewhat contradictory. (See Table 1) For many years falling numbers could be used to argue that students who took A level were becoming increasingly selective in their choice of A level and as such we might expect the number of higher grades to increase, but the increase in students numbers between 2005 and 2007 with a corresponding increase in the number of grades A and B might contradict this. Perhaps the way A level exams are structured so that students can repeatedly retake modules accounts for this? There are perhaps too many variables to use A level results to predict student performance.

<table>
<thead>
<tr>
<th>Year</th>
<th>Students</th>
<th>A</th>
<th>B</th>
<th>A&amp;B</th>
<th>N&amp;U</th>
</tr>
</thead>
<tbody>
<tr>
<td>1992</td>
<td>72,284</td>
<td>20.1%</td>
<td>14.6%</td>
<td>34.7%</td>
<td>24.1%</td>
</tr>
<tr>
<td>1997</td>
<td>68,853</td>
<td>27.8%</td>
<td>20.3%</td>
<td>48.1%</td>
<td>10.9%</td>
</tr>
<tr>
<td>2002</td>
<td>53,940</td>
<td>31.1%</td>
<td>19.0%</td>
<td>58.1%</td>
<td>9.5%</td>
</tr>
<tr>
<td>2005</td>
<td>52,897</td>
<td>40.7%</td>
<td>21.5%</td>
<td>62.2%</td>
<td>2.9%</td>
</tr>
<tr>
<td>2007</td>
<td>60,093</td>
<td>43.5%</td>
<td>21.6%</td>
<td>65.1%</td>
<td>2.0%</td>
</tr>
</tbody>
</table>

Table 1 – A-level results in the UK [1]

Fig 1 – Average score on diagnostic test 1996–2007. The apparent anomaly in 2006 was when Keele introduced a Single Honours Mathematics Course.
Lawson [2] reported the results of performing a diagnostic test on first year students on degree courses with significant mathematical content. He found that a student with a grade B in A’ level mathematics in 1999 scored similarly to a student with grade N (a fail grade) in 1991. At Keele University, we have been running the same diagnostic test on students entering Mathematics. The average score is presented in Fig 1 overleaf.

These results show an interesting periodic variation but it would be difficult to conclude that standards are falling. However, one output of the diagnostic test is a skills profile of each student. Fig 2 shows an aggregate profile for the cohorts 1996-2007. These results certainly show the enormous range of ability in successive cohorts which is disguised by the limited change in the overall average.

2. A level papers

There have been enormous changes in A level over the last 30 years, again with assorted claims that it is now much easier to pass, after all that’s what Table 1 shows. Or does it? Let’s look at some “sample” exam questions from papers selected at “random”.

Fig 3 shows a question from a 1968 CEB paper. This paper contains nine questions; all of which may be attempted but good answers to four such questions are required for a pass. Fig 4 shows a corresponding question from a 2006 OCR mathematics paper that has a total of 72 marks so it is arguable that the same marks are available for both these questions. It would be easy to draw the conclusion that A level is getting easier.

9 (i) Differentiate with respect to \( x \)

(a) \( \log x \), (b) \( \frac{\sqrt{(x^2 - 2)}}{x} \) [4]

simplifying your answers.

(ii) If, at time \( t \) sec., the velocity \( v \) ft/sec. of a particle moving along the axis of \( x \) is given by the formula

\[ v = e^t + 4e^{-2t} \]

and if at time \( t = 0 \) the particle is at the origin, find an expression in terms of \( t \) for its distance from the origin at time \( t \).

Find the time at which the acceleration of the particle will be zero, and the velocity and position of the particle at this instant. [7]

6 (i) Solve the equation \( x^4 - 10x^2 + 25 = 0. \) [4]

(ii) Given that \( y = \frac{2}{5} x^3 - \frac{20}{3} x^3 + 50x + 3 \) find \( \frac{dy}{dx} \) [2]

(iii) Hence find the number of stationary points on the curve \( y = \frac{2}{5} x^3 - \frac{20}{3} x^3 + 50x + 3 \) [2]

However, from the same Examination Board:-

A curve is given by the parametric equations \( x = t^2 - 3, \)
\( y = t(t^2 + 3). \)

Find its \( x, y \) equation, in a form clear of surds and fractions. [2]

Prove that it is symmetrical about the \( x \)-axis. [1]

Show that there are no points on the curve for which \( x < -3. \) [1]

Find \( dy / dx \) in terms of \( t, \) and derive the coordinates of the points on the curve for which \( dy / dx = 0. \) [4]

Sketch the form of the curve. [5]
A curve is given parametrically by the equations
\[ x = 4\cos t, \quad y = 3\sin t, \]
where \(0 \leq t \leq \frac{1}{2} \pi\).

Find \(\frac{dy}{dx}\) in terms of \(t\). \[\text{[3]}\]

Show that the equation of the tangent at the point \(P\), where \(t = p\), is \(3x\cos p + 4y\sin p = 12\). \[\text{[3]}\]

The tangent at \(P\) meets the \(x\)-axis at \(R\) and the \(y\)-axis at \(S\).

Show the area of the triangle ORS is \(\frac{12}{\sin 2p}\). \[\text{[3]}\]

Write down the least possible value of the area of the triangle ORS, and give the corresponding value of \(p\). \[\text{[3]}\]

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It would not be difficult to argue that the 2006 question, shown in Fig 6, is slightly harder than the corresponding question from 1968, shown in Fig 5, since it involves trig functions. Perhaps its not that today’s questions are any easier but it is easier to pass by repeatedly retaking modules. However, it may be just as hard to get a high grade. OK, this needs to be taken with a barrel of salt but it is important we get to the bottom of this.

“Mathematics educators talk endlessly about how important conceptual understanding of mathematics is and whether every student actually reaches this vaulted height. However, just what is ‘conceptual understanding?’”

3. A simple experiment

In 2007, we decided to look at students conceptual and methodological abilities by asking them series of questions (the reason for these questions will be explained later):

1. Tabulate the function \(y(x) = \frac{10}{x} + 40x + 5\), \(\quad 0 < x < 6\)
and use the result to plot this function. Find the slope of the curve at \(x = 0.5\) and comment on the result.

2. Find two positive real numbers that have product 100 and minimum sum.

3. Classify all the turning points of the function \(s(t) = At^{-1} + Bt + C\) where \(A, B\) and \(C\) are all positive.

4. A cylindrical can has radius \(r\) and height \(h\). If the can is to hold 250 cubic units and has minimum surface area, show that the height is twice the radius.

5. The cost of keeping a car is the sum of insurance and running costs. Daily insurance cost is inversely proportional to the age of the car. Daily running cost is the sum of a fixed cost, including tax, and a maintenance cost that is proportional to the age of the car. For what age of car is the total daily cost smallest?

Students were given approximately one hour and told they could answer as many questions as possible.

4. Results

Mathematics educators talk endlessly about how important conceptual understanding of mathematics is and whether every student actually reaches this vaulted height. However, just what is “conceptual understanding?” Devlin gives an interesting discussion on this in an MAA online article [3]. For this experiment, each question was given a marking scheme for “method” and “concept” and marked blind. The results are given below.

As might be expected the conceptual marks are somewhat lower than method marks. This is what Mazur [4] found with a similar experiment involving Physics students in the USA. What is more interesting is that the population appears to be bi-modal. Weaker students, as might be expected, score lower on both scales but at some point the population appears to switch. Perhaps this is an example of an “Aha!” moment when with perseverance suddenly all becomes clear and a student takes a quantum leap to the enlightened group. [5,6] OK, it’s a small population and the results need to be repeatable but wouldn’t it be nice if all that effort we put into the weaker group to give them a leg up actually was beneficial? This “feel-good” effect cannot be underestimated. Liljedahl [7] has investigated the emotional effect of an “Aha!” moment; “the transformative effect on ‘resistant’ students affective domains, creating positive beliefs and attitudes about mathematics as well as their abilities to do mathematics.”

To complete the story Table 2 above shows the average score per question and the score those staff who teach
these students, were expecting them to achieve. As might
be expected depending on individual backgrounds, there
was enormous variation.

5. Comments
So what did the experiment reveal? After the test
students were asked whether they had spotted anything
about the questions. The vast majority said “No.” First let’s
look at the results.

**Question 1.** Many students were able to tabulate the
function and then to plot it. A typical solution, including a
sketch, is shown below in Fig 8. The plotted values looked
linear; so many students assumed that the function is
linear and drew a straight line. Just as many were able to
calculate the slope at $x = 0.5$ and reported it to be zero.
In excess of 30% of students who plotted the function
as linear and also the spotted the zero derivative at 0.5
failed use the turning point to correct their graph! Bite size
mathematics at its worst!

**Question 2.** Despite the fact that students were told they
had to use the calculus they simply used trial and error. Few
students were able to solve this question as asked even
though staff teaching the course fully expected them to be
able to do so. (See Table 2)

**Question 3.** Nearly all students were able to differentiate
this function and show that there are two turning points,
$\pm \sqrt{B/A}$.

**Question 4.** Many students were able to write down the
formula for the volume and surface area of the cylinder. Few
were able to solve this constrained optimization problem of
minimising $S = 2\pi r^2 + 2\pi rh$ subject to $V = \pi r^2 h$ by
eliminating the constraint to maximise $S = 2\pi r^2 + 2V/r$.
Those that did were unable to show that the height is twice
the radius.

“Students are undeniably driven
by assessment but they seem
unable to put together a significant
mathematical argument.”

**Question 5.** Few students were able to get started on this
problem. Some were able to write down an expression
for the cost as $C(t) = \frac{A}{t} + Bt + D$ but were then unable
to optimize it even if they were able to solve Question
3 completely!

Most students at Keele area admitted to a dual honours
program and have Mathematics A level at grade C and
above. They may not be typical but I would suggest that the
above results would be typical at many universities in the
UK. This has been confirmed when this test was used to a
number of other institutions. Furthermore, results from as
far a field as comparative courses in the US, Australia and
Turkey give very similar results

So what was the point of the experiment?
Looking a little closer at the questions – they are in fact
all the same, only the level of conceptualization changes.
Students were able to handle the algebraic form of the
Question 3, but unable to tackle the same problem as
posed in the Questions 4 and 5. What’s more even having
solved Question 3 they could not even start the same
question posed non-algebraically.

6. Conclusions.
This study indicates that students seem to be able to
do bite-size, piecemeal mathematics, but seem unable
to see the bigger picture. They are undeniably driven
by assessment but they seem unable to put together
a significant mathematical argument. However, if the
“Aha!” moment produced by this experiment is a real
phenomenon then perhaps there is a justification for
spending so much time with struggling students in the
expectation that at some point when suddenly everything
becomes clear:
“I had been working on the problem for a long time without any progress. Then suddenly I knew the solution, I understood, everything made sense. It seemed like it just CLICKED!”

Student quote from Liljedahl [7]

How many times have you seen this happen? Even after many years teaching, I still get a warm glow when I see it take place.

This experiment needs every possible caveat, the sample is too small and perhaps unrepresentative, and the results need repeating and further analysing. It’s a small sample and may not be significant; the only way to find out is to repeat the experiment. So, I’m looking for volunteers – I’ll send you the test in electronic form, I’ll even mark it!

References