Mathematicians and mathematics educators have an uneasy relationship. To provide a forum at which both mathematics researchers teaching undergraduate students, and education researchers could share a platform and discuss issues of mutual concern, the MSOR Network has organized two workshops at the University of Warwick. The first took place on 19 March 2007, with the title “Mathematicians and Educationalists: can we collaborate?” This was a lively meeting, at which it was agreed we should arrange a follow-up meeting. This took place on 19th March 2008 with the title “Mathematicians and Educationalists: how can we collaborate?” The speakers: Peter Saunders, David Tall, Alice Rogers, Alexandre Borovik and John Mason again provided us with a very interesting day of contrasting views and concerns. The article which follows is by Prof. John Mason, and expands on the talk which he gave at this meeting.

Abstract

Participants will be invited to experience the use of mathematical phenomena to stimulate interest and to encounter core-awarenesses which lie behind mathematical topics, and to develop richly interconnected example-spaces. These experiences will be related to a four-fold structure of the human psyche which can inform choices of pedagogical strategies and didactic tactics. Ancient psychology suggests a four-fold structure of the human psyche, often referred to in the west as cognition, affect, enaction and will-intention. Recasting these as awareness, emotion, behaviour, and attention, I will suggest ways in which learners can be provoked to harness their emotions towards educating their awareness and training their behaviour through re-structuring their attention.

Notes in retrospect

The following notes are a reconstruction of some of the proposals and suggestions made in the session, together with some that I did not have time to include.

Didactic transformation, didactic transposition and didactic tension

Alexandre Bokovic introduced the didactic transformation as the process of turning mathematical ideas into teaching materials. He traced the notion back to Auguste Comte, and asked whether there was any support for this process of transformation.

John Dewey [4] used the phrase psychologising the subject matter to refer to the process of making ideas accessible to students, and this was reformulated by Lee Shulman [17] who addressed the question of what knowledge teachers need in
order to do this. He proposed several 'types of knowledge,' including that of pedagogic subject knowledge.

However, Yves Chevallard [3] coined the expression transposition didactique to refer to the inevitable alteration made when an expert prepares subject matter for teaching. I like to cast this as:

“Expert awareness is transposed or transformed into instructions in behaviour.”

In other words, an expert converts their deep appreciation and understanding of a topic into a sequence of instructions for students to follow (tasks to do, definitions, examples, theorems, proofs to follow, etc.). But as Chevallard points out, what the teacher is trying to do is to construct tasks for students which, it is hoped, will reproduce for them what the teacher has experienced. The lovely examples that Alexandre showed us fit this description all too well, because the result of his deep insights into the subject matter is to have experienced (such as preparation for tensors) results in ‘things for students to do, in a notation that is consistent with what they will be needing the ideas for’. As long as the students are able to transform the tasks they are given, into appreciation and understanding, all will be well. The notation and approach to the subject matter taken may indeed assist this, but ultimately the students always have to take the final step to re-transform their activity into understanding.

A related notion is the didactic tension, a rephrasing of implications of the expression didactic contract coined by Guy Brousseau [2]:

“The more clearly the teacher indicates the behaviour sought, required or expected, the easier it is for learners to display that behaviour without generating it from themselves.”

In other words, when we set assessment tasks we specify the kind of behaviour required to ‘pass’, which is taken to be evidence of ‘learning’. Some students transform what they are exposed to, and come to understand and appreciate the underlying ideas. This enables them to reproduce for themselves, to reconstruct when needed any details they have forgotten, and so to resolve multi-stage problems successfully. Many students quickly work out what behaviour is expected, and reproduce that, as in reproducing text from lectures complete with typing errors! For Chevallard, there seems to be an inevitability, mirrored in this somewhat pessimistic didactic tension.

What might it mean to ‘learn’ a mathematical topic?

I proposed some possible responses:

- Gain facility with techniques?
- Develop fluency in use of technical terms to express oneself?
- Gain access to a richly interconnected web of examples?

- Appreciate the origins and range of applications of the topic?
- Be aware of classic obstacles and misconceptions?

I suggested that absence of any of these meant an imperfect appreciation of a topic. In order to test for understanding, we probe students by giving them tasks. However, what is testable is behaviour, in response to probes. All attempts to test ‘understanding’ amount to probing the student and gauging from their response whether they understand, as expressed above. For me, this is captured in the analogy:

“Wounds are to a patient as assessment is to students.”

In other words, you only learn about the health of the body when it is put under pressure in some way and has to respond. This is what we do to students, even when we give them open choice of topic for a project: their task is to display the behaviour being looked for. One good way to do this is to act mathematically; another, which most students resort to at some time, is to try to work out what is being looked for and then to display that.

I suggest that it is possible to probe understanding a bit more deeply, as well as to facilitate understanding and appreciation of mathematics by working with students on the question of ‘what does a student need to do in order to learn mathematics’.

This question is too big to address fully and comprehensively here, or in a 40 minute presentation. My aim was to offer one or two ideas towards that, and to indicate that there is support in the mathematics education literature for the design of courses, lectures and tasks for students which could contribute to prompting students to study mathematics effectively and efficiently. The five questions listed above are taken from a framework articulated for precisely this purpose [8].

I mentioned that there is a considerable literature on what it means to learn mathematics. It is inappropriate to attempt to review that literature here, but see Mason & Johnston-Wilder [12] for access to some of it.

Psychologising subject matter

My main aim was to show how surprising phenomena can be used to initiate student engagement with a topic. With more time I would have expanded this to the use of phenomena to sustain activity, and to promote integration and condensation as understanding. All of this is informed by the framework which I introduced near the end of the session. I wanted particularly to draw attention to the notion of core awarenesses: specific awarenesses which underpin or lie at the core of appreciation of a topic.

Tilted can

I demonstrated the tilted can phenomenon [7, 14] which surprised at least some of the audience. My purpose was
to demonstrate how it is possible to provoke learners into action by providing a surprising phenomenon which can be accounted for using the proposed mathematical topic. More specifically, I wanted to use the can to ask about what it means to understand. Is it sufficient to say a few words about centre of gravity to display understanding? If not, is it sufficient to be able to outline how you might model it mathematically (assume the inside of the can is cylindrical, calculate the position of the centre of gravity of whatever liquid is inside, using an integration)? If not, do you require someone to perform the integration (formulating the vertical sections is not entirely trivial!).

I hoped that this would provide a direct experience of various forms of ‘understanding and accounting for a phenomenon’.

Reading graphs

For many students, it seems that a graph is the end product of drawing or sketching, rather than the starting point for recognising relationships and perceiving properties associated with a conjecture, definition or theorem. In the 1970s, we used cobweb diagrams to display iterations but it turned out that many students had no idea what was going on. There is a dual perception of graphs which is vital: as objects and as relationships. It is really useful to imagine a point running along the $x$-axis and attending to the corresponding point running along the curve, so that the curve is made up of points each of which has coordinates. To develop this awareness, I find the following animation provocative (Fig 2).

After watching the film (Fig 2), students are asked to describe what they are seeing, and then to work out from coordinates what the locus of the end point is.

Once students get the hang of tracking coordinates, they get a visual sense of what composition of functions is about. They can explore and exploit this by drawing a diagram to display the composition of three functions and to see directly and geometrically why composition is associative. Furthermore, they have developed a behaviour they can use for interpreting other graphs they come across. For many it may be necessary to draw attention to the fact that the two functions are intended to be seen as generic. Indeed, if they can say what it is about functions that makes the construction viable (well defined) then they display some understanding of what functions are about!

Tangent power

I discovered that many students are unclear about what happens to the functions they sketch as $x$ gets very large. This may be due to the graph ‘disappearing off the top of the page’. To work on an appropriate awareness, I pose the tangent-power task (Fig 3):

Many people start with a point and sketch tangents, but then realise that to get at regions it might be more helpful to track a tangent running along the curve. This gives them an enactive appreciation of the movement of the tangent, and insight into the meaning of the second derivative. Further exploration could be undertaken to account for
the emergent phenomenon that the regions all have odd tangent power in this case: under what circumstances would they have even tangent power ... and so the engagement in mathematical activity, and engagement with mathematical thinking, is sustained and stimulated. At the same time, their awareness of the meaning of second derivatives is enriched. As a by-product, they work on the issue of what happens to tangents for large $|x|$.

Promoting learning

In common with most educational thinkers to whose thinking we have access, and in particular with David Tall's presentation, teaching involves three principal activities:

1. Initiating action;
2. Sustaining activity; and,
3. Promoting learning from experience.

In each of these phases, what a teacher can do for students is direct their attention and display mathematical thinking themselves. Students can be stimulated by tasks, which engage them, to undertake mathematical activity. The tasks have to be appropriately challenging (neither too far out of reach, nor too simple). This means tasks that call upon students to use familiar actions but in unfamiliar ways to tackle unfamiliar problems. There is a huge literature on related issues to do with motivation. Here I was trying to illustrate how students can be prompted to harness the emotional energy which is released when there is a surprising phenomenon to be accounted for.

It is vital that students sustain their activity so that they make substantial contact with core concepts, core awarenesses which underpin the topic. They also need to become aware of the modifications they made to the actions they used so that these too can become routine, so that they begin to become aware of how they have used and developed their natural powers to think mathematically, of the presence of ubiquitous mathematical themes, and of the effectiveness of heuristics that have been used or suggested.

One way to do this is to get learners to construct their own examples of concepts, and to illustrate theorems or to demonstrate the necessity of conditions in those theorems. Weakened conditions. The richer the web of examples and example-construction techniques to which students have been exposed, the richer their sense of understanding of the topic.

More on core awarenesses

Eigen Directions

I used a dynamic geometry package to provide a visual image of what it means to specify a mapping from a standard basis in a vector space to two linearly independent vectors in that same space, and to extend it to a linear transformation. Geometrically, you use the co-ordinates of a vector $v$ specified in the standard basis, as the co-ordinates of the image $w$ but relative to the image basis $(f'_1, f'_2)$. When you drag $v$ about the screen you see directly linearity and the effect of scaling $v$. You also see that sometimes $v$ and people's examples is much less effective, because there is a tendency to 'monster bar' [5] rather than learn from counter-examples and so-called 'pathological examples'.

Activity is simply activity. Learning from experience is not a necessary result of having experience, or put another way: "One thing we don't seem to learn from experience is that we rarely learn from experience alone."

One way to learn from experience is to consolidate understanding by undertaking a task which requires considerable understanding. For example, Alexandre's task of finding a basis for the intersection of two vector subspaces of a vector space, given bases of those subspaces, or the task of characterising those matrices which can be used to represent a non-singular linear transformation of one vector space onto another [7, 14]. Another way is to construct a narrative which makes use of the technical terms in a meaningful way. Another is to explain to someone else.

But learning from experience is more than consolidating understanding by accounting for phenomena using the topic, or by exploiting their understanding to explore a complex task. George Pólya [16] proposed four phases of problem solving, the fourth of which is looking back. Unfortunately, it is seldom exploited. A significant form of attention directing performed by teachers is to remind students or to highlight what they have done that is of significance in the way of use of their powers, of the use of heuristics, and encounters with mathematical themes. By labelling those contacts, learners can be supported in having those actions come to mind in the future [9]. For example, getting students to construct a narrative about what they have done (mathematically), so that they are encouraged to move from a 'sense of' what they were doing to an articulation which will eventually become automatically triggered. Similarly, students can be encouraged to construct their own examples of important constructs to illustrate theorems and definitions, to meet extra conditions involving important concepts, and to justify the presence of hypotheses in theorems through counter-examples to weakened conditions. The richer the web of examples and example-construction techniques to which students have access, the richer their sense of understanding of the topic.
w come closer to each other and even overlap. This raises questions about eigen directions and eigen values. But it also raises other questions not usually asked, such as where the angle between v and w is a minimum (when there are no eigen directions) and a maximum. Furthermore, holding $f_1$ fixed, different positions of $f_2$ result either in the existence or non-existence of eigen directions, so the boundary between these regions must be of interest. Finally, the image of a circle is (in general) an ellipse, but what is the relation between the axes of the ellipse and the eigen directions (when they exist)? Students have an opportunity to distinguish between the image of a set of points, and the set of images of points: the points you attend to may not be the images of the points you think they come from!

The dynamic display was designed to afford access to a visual version of the core awareness of what eigen directions are about, and what happens when they don't exist. A by-product is a phenomenon to intrigue and initiate the topic, and a phenomenon to explore in order to consolidate that awareness. In his presentation, Peter Saunders was making a plea for more time for students to digest mathematical ideas and mathematical ways of thinking in their first term or first year. By digesting, I presume he includes such things as exploring further implications, making contact with the natural powers they use, becoming aware of pervasive mathematical themes and heuristics, and becoming aware of how to study mathematics [6, 8].

Group actions

I had prepared an example of displaying group actions in such a way that students would be less likely to confuse the objects for the actions which constitute the group (of actions), and at the same time, in exploring the phenomenon, encounter subgroups, cosets and the orbit-stabilizer theorem as a natural consequence of accounting for the phenomenon. However, there was insufficient time to do this.

Structure of a topic

The preceding examples were intended by way of an introduction to a framework for thinking about the constituents of a mathematical topic that I have used for over 25 years, and which formed the core of some booklets written in 1984 for secondary teachers retraining as mathematics teachers. They are currently still used in some PGCE courses! Three strands are brought to the fore, each with two aspects. The roots of the framework lie in the Upanishads, and the analogy between a chariot and the human psyche. My suggestion is that in preparing to teach a mathematical topic it behoves us to take account of, indeed to mirror the structure of the human psyche. I have expounded upon it elsewhere [11, 12, 14, 15]. Suffice it to say here that the three strands correspond to awareness, emotion and behaviour, and that the questions about learning with which I began, reflect the principal import of the strands. I would put forward this three-stranded framework as a contribution to Alexandre’s request for suggestions as to how to design a course, a lecture or a task; that is, what aspects to pay attention to when considering choices of topic, topic-organisation, topic presentation, pedagogic strategies, didactic tactics, and student activity, from initiating it, through sustaining it, to consolidating and assessing it.

Attention

I mentioned in passing in the session that one way to make sense of the difficulties students have in grasping what we show and tell them, and particularly in passing from informal ‘sense-of’ concepts and theorems to a more formal presentation, is in terms of attention. Students are not always attending to the same thing as the lecturer, but even if they are attending to the same thing, they may be attending in subtly different ways. I distinguish five forms of attention [10, 12], supported by a wide gamut of literature including gestalt psychology, van Hiele’s geometric ‘levels’, and ancient psychology:

1. Holding wholes (gazing);
2. Discerning details;
3. Recognising relationships;
4. Perceiving properties; and,
5. Reasoning on the basis of agreed properties.

A great deal could be said about this. Suffice it to say here that whereas a lecturer may be seeing a diagram or a mathematical expression as an instance of something more general, that is, as instances of perceived properties, students are more likely to be aware only of the specific in front of them, and at best to be recognising relationships in the particular. This applies no matter how general or how particular the diagram or the expression. To see through a particular to a generality which it instantiates requires a shift of attention which may not be available to some students at the required moment. They may take longer to discern pertinent details and to recognise pertinent relationships than does the expert. Consequently, students find ‘reasoning on the basis of agreed properties,’ that is, constructing proofs of statements of things they thought they knew, as in analysis, or of things they don’t yet fully appreciate, as in linear algebra and group theory, really difficult.

Collaboration possibilities

Possible avenues for fruitful collaboration between mathematicians and mathematics educators include:

1. Discerning and articulating core awarenesses underpinning topics, particularly topics that cause many students to struggle; and,
2. Development of a shared vocabulary for pedagogic strategies (which can be used in many different
contexts) and didactic tactics (which apply to specific topics or awareness).

I suggest that these happen through dialogue or multilogue, that is through negotiation in particular instances. There are no theorems, no sweeping results which provide algorithms for making pedagogic decisions. Teaching effectiveness arises from sensitivity to both students and subject matter and is situation-dependent.

One thing that is necessary for collaboration is mutual respect, and that involves an appreciation that mathematics and mathematics education have fundamentally different objects of study. Whereas mathematics proceeds by justifying assertions on the basis of accepted forms of reasoning, mathematics education proceeds by trying to articulate insights and awarenesses about the functioning of human beings, whether individually or in groups. As such, there are no theorems in mathematics education. Some people acknowledge justification in the form of statistical evidence, but very few people would undertake a change of behaviour which did not in some way 'make sense' to them, which did not in some way fit with their perceptions and approach. For example, there is no 'best way' to teach a topic, because the range of factors which influence student learning is so great, ranging from the social effects of peer groups and expectations, through their own psychological dispositions and sensitivities, to the institutional requirements and the nature of the relationship between student(s) and teacher, not to say the structure of the mathematical topics involved.

Final remark

Attention was drawn in discussion to the need for mathematics-specific materials for staff development, because generic staff development generally misses the mark so completely. The publication Mason [2] arose from precisely this concern, and is a development of materials originally produced as a contribution to a generic course, designed specifically for mathematicians.

References