In this paper I describe the rationale behind the construction of a set of resources designed to help students better understand the proofs in an undergraduate real analysis module.

1. Theoretical background

Analysis is difficult to learn and difficult to teach, not least because it is usually one of the first modules in which students must engage with a large number of abstract proofs. Research in mathematics education indicates that students find it difficult to construct proofs [1,2], that they often behave as though their beliefs about the nature of proof were different from those of expert mathematicians [3,4], that there is often a distinction between what they find personally convincing and what they believe is acceptable as a proof [5], and that their ability to distinguish valid from invalid proofs is unreliable though may improve with ongoing mathematical education [6]. Further, that undergraduates are often unaware of the status of definitions within mathematics [7], which will obviously affect their ability to recognise the use of these within proofs. In real analysis this last difficulty is compounded by the centrality of the limit concept, which has a logically complex definition involving three nested quantifiers.

Much of this cited research focuses on proof construction or – less usually – on proof validation. Very little has focused on proof comprehension, which is arguably a much more common task for students. Comprehension questions have been recommended as an alternative to standard examination questions: Conradie & Frith [8] described their own experience of giving a standard proof that \( \sqrt{20} \) is irrational and asking questions about it, including those below. They argued that this gives a better indication of students' understanding than standard questions that can be answered via rote memorisation.

- What method of proof is used here?
- When is a real number irrational?
- Why does \( 5k^2 = 4n^2 \) imply that 5 is a factor of \( n^2 \)?
- Which assumption is contradicted, and how does the theorem follow from this?

Some of these questions (a. and f.) are about the structure of the proof and how this structure relates to the theorem statement. In this sense they echo Selden & Selden's [9] discussion of proof frameworks, and the necessity of being able to "unpack the logic" of a mathematical statement if one is to establish whether a given argument proves that statement. Selden & Selden found that students were often not able to do this unpacking effectively. The other questions ((c) and (e)) require what Weber
& Alcock [10] called inferring warrants, after Toulmin’s [11] scheme for analysing arguments. In this scheme, an argument is broken down into components including the data, the conclusion and the warrant. The conclusion is the assertion about which one wishes to convince an audience, the data is evidence put forth, and the warrant is the reason that the data support the conclusion. Weber & Alcock argued that warrants are often implicit in mathematical arguments, so must be inferred by a reader. They gave an illustration of the process of inferring warrants for a proof in real analysis, which begins (proof taken from Abbot, [12]; separation into numbered lines by the authors):

Statement: If a sequence is monotone and bounded, then it converges.

Proof:
1. Let \((a_n)\) be monotone and bounded.
2. To prove \((a_n)\) converges using the definition of convergence, we are going to need a candidate for the limit.
3. Let’s assume that the sequence is increasing (the decreasing case is handled similarly) and consider the set of points \([a_n : n \in \mathbb{N}]\).
4. By assumption, this set is bounded,
5. so, we can let \(s = \sup(a_n : n \in \mathbb{N})\).
6. It seems reasonable to claim that \(\lim(a_n) = s\) [a number line diagram is included at this point].
7. To prove this, let \(\varepsilon > 0\).
8. Because \(s\) is the least upper bound of \([a_n : n \in \mathbb{N}]\),
\(s - \varepsilon\) not an upper bound…

Weber & Alcock’s gave the following breakdowns for lines 5 and 8:

“Line 5: In this line, the data is the fact that the set is bounded. The word “so” indicates that this data appeared in the immediately preceding phrase. The conclusion is that the set has a supremum and for a warrant one must infer that any subset of the reals that is bounded above has a supremum in the reals, i.e. that the reals are complete.

Line 8: In this line, the data is the fact that \(s - \varepsilon\) is greater than 0 and that \(s\) is the supremum of the set. The fact that the latter is data is explicitly stated via the “because” clause. The conclusion is that \(s - \varepsilon\) is not an upper bound. The warrant is that there can be no upper bound for the set that is less than the supremum – the use of the alternative terminology “least upper bound” serves to emphasize that this is the warrant being used. One may also note that a sub-argument could be used as backing for this warrant. This would have data that \(\varepsilon\) is greater than 0, warrant that if \(x > 0\) then \(s - x < s\) and conclusion that \(s - \varepsilon < s\).”

(Weber & Alcock, ibid., p.38)

Studies reported in both Alcock & Weber [13] and Selden & Selden [14] indicate that students are not always able to reliably infer and evaluate warrants when validating a proof, but that opportunities to reflect upon their reasoning may improve performance.

These two types of proof comprehension question correspond well with some of those used in a study by Lin & Yang [15,16], which did focus specifically on proof comprehension (in geometry). Lin & Yang classified their proof comprehension questions into six categories: basic knowledge, logical status, integration or summarization, generality, application or extension and appreciation or evaluation. Their logical status questions were similar in nature to warrant-inferring questions. Their integration or summarization questions were similar to larger scale structural questions.

While proof comprehension remain under-researched, it seems likely that it involves asking and answering questions like those of the two types discussed across these studies. However, this process is not usually demonstrated to students. While presenting a proof, a lecturer will typically give explanation about implicit warrants and overall structure, but most of this will be lost when the material is codified in static lecture notes. Most lecturers do not give explicit instruction on how a student might reconstruct this information during independent study. The e-Proofs described in the next section constitute a resource designed to address this.

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2. Structure of e-Proofs

I designed and constructed the e-Proofs with the support of an Academic Practice Award from Loughborough University, where they have now been used in the analysis module. This module was taken by approximately 140 students, and delivered via two 50-minute lectures per week for an 11-week term. The students were then split into three groups for tutorials, in which they worked in groups on problems related to recent and forthcoming material. The course covered standard material on continuity, differentiability and Riemann integrability. “Gappy” notes were used so that students had more opportunity for interaction within lectures than might be typical, but nonetheless the majority of the students’ work...
in understanding the theorems and proofs would probably take place outside lectures.

The e-Proofs were made available to the students via the university’s virtual learning environment (VLE). Each e-Proof is presented as a sequence of screens and accompanying audio files, and each comes in three versions:

1. A basic version in which the proof appears one line at a time and the audio commentary reads that line;
2. A line-by-line version in which the whole proof is visible but greyed out, with each screen showing one line (or part of a line) fully visible, arrows and boxes indicating why that line is valid and its relationships to the other parts of the proof, and the audio providing explanation; and,
3. A chunk version, in which the whole proof is visible but greyed out, with each screen showing several lines fully visible, a box indicating what that section achieves, and the audio describing this in a little more detail.

In the majority of cases, students were given a printed copy of the whole proof, invited to spend a few minutes trying to understand it, and then shown the line-by-line and chunk versions. After this, the e-Proof was made available via the VLE. The basic version was designed to function as a rerun of a standard lecture, perhaps of particular use to students whose first language is not English. The other versions, and the rationales behind the design of each, are described in the next sections.

3. Line-by-line version

The line-by-line version of each e-Proof is designed to demonstrate to students the process of inferring warrants. Taking one line (or part of a line) at a time, it indicates what part of the assumptions or earlier reasoning forms the data that allow us to reach that line as a conclusion. In cases in which the warrant must be inferred by reference to previous results, these are described in the audio commentary. This commentary also goes some way toward promoting focus on the structure of the proof by referring forward to later lines to indicate why we are setting up assumptions and notation in certain ways.

A screen from a line-by-line version is shown in Fig 1. The accompanying audio says:

“In the first line, we state our assumption that \( f \) and \( g \) are continuous at \( a \), which corresponds to the premise of our theorem. We also let epsilon greater than zero be arbitrary, because we want to show that \( fg \) satisfies the definition of continuity at \( a \), which we will achieve by the end of the proof. Doing so involves showing that something is true for all epsilon greater than zero, so choosing an arbitrary epsilon means that all our reasoning from now on will apply to any appropriate value.”

4. Chunk version

The chunk version of each e-Proof is designed to demonstrate to students the process of breaking a proof into chunks, each of which can be thought of as a separate and somewhat internally coherent stage in the proof. This is related to Selden & Selden’s [10] discussion, in that it should facilitate recognition of the way in which the structure of the proof is related to the structure of the theorem statement. It is also designed to help students recognise that a proof might be easier to remember as a sequence of four major steps than as thirteen separate lines.

A screen from a chunk version is shown in Fig 2. The accompanying audio says:

“In the third chunk, we set up an overall delta value, and put together the information from the second chunk to show that if the modulus of \( x \) minus \( a \) is less than this delta, then our original modulus expression is less than epsilon.”
5. Using the e-Proofs

As stated above, I used the e-Proofs for the first time in 2008. I collected usage data via the VLE and gathered qualitative student feedback once the module was completed. This is reported in Alcock [17]. Here I offer some comments on the effect that the resource has had on my own experience of the module. First, I have found the e-Proofs satisfying to use as a lecturer. They give me greater than usual confidence that what I am seeing when I point out reasoning and structure in proofs is actually visible to the students. Second, my impression is that using the e-Proofs serves to highlight how much work might go into understanding a proof, so that the students were aware of this from the beginning of the course. Third, using the e-Proofs has definitely increased the number of comments I make about the process of understanding proofs.

The current versions of the e-Proofs are far from perfect, however, and I am collaborating with the university’s e-Learning team on a JISC-funded project called ExpOUND (see http://expound.lboro.ac.uk) to develop a web-based tool for lecturers to use in constructing their own e-Proofs that:

1. offer better time-linking of audio commentary with appearance of arrows etc., and,
2. incorporate the progressive construction of appropriate diagrams. We plan eventually to make these more widely available.

Of course, the positive impressions cited above are personal and do not constitute empirical evidence that the e-Proofs are an effective learning resource. With the support of MSOR Network funding, colleagues and I are currently conducting research to investigate whether study of an e-Proof does result in better proof comprehension than an equal amount of time studying an ordinary written proof, and whether students can transfer the skills of inferring warrants and breaking down structure to other proofs for themselves.

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References