What is essential in mathematics education? In my opinion, the mathematics curriculum must provide the average student with the tools to effectively address significant problems in that student's life. This article concerns an approach to teaching mathematics that facilitates the student seeing the connection between the process of solving non-routine mathematics problems and solving significant problems in the student's life. The approach is embedded in a curriculum of non-routine problems field-tested from 1984 to 1995 in a secondary school setting, and later applied in an undergraduate mathematics course in problem-solving. In this article, I will define a non-routine problem; discuss a few examples of non-routine problems, including some that facilitate the student seeing the connection between the process of solving non-routine mathematics problems and solving significant problems in the student's life; and suggest guidelines for implementing a curriculum of non-routine problems.

Definition of a Non-routine Problem

For instructional purposes, a non-routine problem at an appropriate level of difficulty has the following characteristics [10]: (a) The problem requires three steps to complete: problem recognition and orientation, trying something, and persistence; (b) the problem allows for various solutions and requires students to evaluate a variety of potential strategies; and (c) a good solution requires the student to use one or more mathematical problem-solving strategies such as finding a pattern and generalizing, generating and organizing data, or reducing a problem to an easier equivalent problem. Though the quality of solutions will vary, students will be able to confront the problem and generate a solution consistent with their ability and efforts.

In other works [7, 8, 9, 11], I have discussed the theoretical and pedagogical basis for this definition, including arguing that the definition of the three steps of problem-solving is consistent with other definitions of the steps of mathematical problem-solving [15] and that a properly structured curriculum of non-routine problems is consistent with constructivist theory [2, 5, 16]. Previous research [8, 9] supports the hypothesis that: (a) The ability to solve non-routine problems is an essential skill in mathematics education in the sense that these skills can be applied by students to solve significant problems in their life; (b) even most above average ability students in mathematics need significant instruction in this objective; (c) most students need an integrative approach emphasizing problems from a variety of fields, including problems meaningful to them, to insure transfer to significant problems in their personal life; and (d) successful instruction is feasible within a curriculum consistent with the content requirements of traditional content-focused courses. In addition,
a curriculum of non-routine problems can be easily integrated into other contemporary approaches to curriculum such as project-based learning [4], creating communities of mathematical inquiry [14], rich tasks [13], or WebQuests [12].

Sample Non-routine Problems

In this section, I will discuss a few sample non-routine problems, some that deal directly with mathematics content, and others that are integrative. This section prepares the reader for the next section concerning implementing a curriculum of non-routine problems.

Mathematics problems

Initially, most students need a careful introduction to the three steps. Therefore, I give initial problems that concretely give most students a successful experience with the three steps. For example, the problem “Expensive Tape” is a non-routine problem that can be used early in the curriculum to introduce the students to the process of solving a non-routine problem. Students complete the following task: determine three boxes that hold the most volume, given that the box is taped with exactly 30 inches of expensive tape, covering one length and one girth (two widths and heights) of the box, as illustrated in Fig 1.

Two of the three dimensions (length, width, and height) must be whole numbers; the reason for this restriction is clarified in the discussion of the solution. For students with little experience solving non-routine problems, I demonstrate a procedure to generate data and encourage the students to generate a good amount of data and to look for two patterns that will help them solve the problem. When students generate data they typically notice two patterns in the data. Before reading on, see if you can spot the two patterns in the data shown in Table 1.

The two patterns illustrated above are: Girths closest to squares yield higher volumes, and lengths closer to ten inches tend to yield higher volumes. Most groups notice these two patterns, thereby finding good solutions to the problem, such as: 10" by 5" by 5" (250 cubic inches); 8" by 6" by 5" (240 cubic inches); and 12" by 5" by 4" (240 cubic inches). Students that persist realize that 9" and 11" lengths should provide better solutions than 8" and 12" lengths (being closer to 10"), and discover the following two solutions, each of which includes one fraction: 9" by 5" by 5 1/2" and 11" by 5" by 4 1/2" (both with a volume of 247.5 cubic inches). Many students initially assume that lengths of 9" and 11" will not work because the sum of the width and height will not be a whole number.

The processing usually helps students not only see how trying something results in uncovering patterns that make the problem easier, but also notice how persistence can lead to the best three solutions. Given the help described, most groups of students are successful with this problem, meaning not necessarily that all the groups find the best solution, but rather that most of the students are engaged in working on a problem at the appropriate level of difficulty and participate in the processing of the problem; a good indication that their problem-solving abilities will naturally improve [3].

\[ y = e^x - \sin x \] is an example of a non-routine problem that is appropriate for more advanced students with experience solving non-routine problems. The task is to find the fiftieth root less than 0 for the equation \[ y = e^x - \sin x \]. You cannot solve the equation for \( x \) through normal algebraic manipulation, rather the student needs to generate data using a graphing calculator and look for patterns. In working with calculus students with experience solving non-routine problems, a typical student solution is as follows: (a) Problem recognition: the student realizes the problem is not solvable by algebraic manipulation, sees the need to generate data, and has some confidence that trying something and persisting will lead to a solution; (b) trying something: the student uses a graphing calculator to generate the first few roots and looks for patterns, noticing a fairly linear decrease in the value of the roots;
and (c) persistence: in trying to make sense of the pattern, the student realizes that the roots change by a value approaching π and realizes that ex contributes close to zero to the value of ex - sin x as x decreases in value; therefore, the appropriate root of y = - sin x (i.e., - 50) is an excellent approximation of the actual root. When I have tried this problem with Calculus students with experience solving non-routine problems, approximately seventy-five percent of the students found an excellent solution.

**Integrative problems**

A major objective of the curriculum is that students will apply the three steps to solve meaningful problems that might not appear to be mathematical in nature, but require the three steps for a good solution. A representative problem of this type is “Planning a Trip,” which requires groups to plan an actual class trip that (a) will increase the class’ appreciation of diversity, (b) is interesting and enjoyable from the students’ point of view, and (c) is inexpensive. In the field-testing, one high school class planned a trip to Boston. Their research uncovered many good ideas including: (a) visiting a variety of relevant museums such as the Museum of the National Center of Afro-American Artists, Cambridge Multicultural Arts Center, and the African Meeting House; (b) seeing relevant sights in culturally rich sections of Boston; and (c) interacting with Cultural Survival, an organization that focuses on helping indigenous cultures survive. Perhaps the most interesting idea was to plan an afternoon with a group of inner city students trained to be peer leaders for an after school program. The idea was to share skills; we would lead a problem-solving session and they would lead a community building activity.

How is this problem connected with the mathematics curriculum? First, the students see how the three steps of solving non-routine problems are useful in solving this practical problem. Students might demonstrate the following type of evidence of applying the three steps: (a) Problem recognition and orientation: Students might realize that some of the criteria for evaluating their project are contradictory, and part of a good solution will be finding possibilities that satisfy all three criteria. In addition, they might realize the need to gather data from classmates rather than relying solely on their own ideas. (b) Trying something: Students might brainstorm possible sources for ideas and decide which sources to actually investigate. They might decide to develop a questionnaire in order to gather data. (c) Persistence: Students might realize that there is a need for additional data to finish the problem well. For example, most questionnaires give some clear data, but also generate some data that suggests the need to ask follow-up questions.

In addition, students many times improve in specific mathematical content areas. For example, in “Planning a Trip,” after the groups report out concerning their ideas, there might be ten suggestions for portions of the trip. The students can be given the task of developing a method to evaluate quickly, yet accurately, which of the ideas are most liked by the class. Briefly, for example, students might discover that one vote per student is typically not enough data to draw conclusions, and that three votes per student will basically generate clear results. Students also get a sense of when some additional data is needed to clarify the results (i.e., there is a need for persistence). After a few problems of this type, most students develop a sense of how to quantitatively generate the right amount of data to clarify a question of this type quickly, yet accurately. My experience is that students who have appropriate experience solving this type of problem are surprised at the quality of results that they achieve, and begin to appreciate the power of the three steps.

When students demonstrate a reasonable understanding of solving non-routine problems, I require them to formulate and solve meaningful non-routine problems in their own life. For example, one high school vocational student of poor mathematics ability had a lawn care business with a few clients with large lawns. He had difficulty organizing the business and was ready to give it up. His problem was to effectively organize the business. To gather information, he interviewed his clients and three lawn care professionals. His initial conclusions included (a) plan a schedule to complete the lawns in four days, allowing for bad weather and other complications; and (b) schedule your largest lawn early in the week. Based on these conclusions and other data, he devised a tentative plan that he checked with his clients and a fourth lawn care professional. He successfully implemented the plan and was very satisfied with the results. This is one example of many that indicates the power of a curriculum of non-routine problems. This student, considered below average in mathematics ability, was able to define a significant problem in his life and apply the steps a mathematician uses to solve a difficult problem to effectively solve his problem. I am not claiming that this student (or similar examples) has the same abilities as a student with excellent mathematics ability, but rather that he had a good understanding of the process of problem-solving and could apply that understanding to solve significant problems in his life.

**Implementing a Curriculum of Non-routine Problems**

In this section, I will discuss what is meant by a curriculum of non-routine problems. Before giving one example of a sequence of non-routine problems, I will briefly describe some of the general characteristics of a curriculum of non-routine problems, as well as some guidelines in implementing such a curriculum -- topics more thoroughly discussed in previous works [9, 10]. I recommend a curriculum consisting of ten to fifteen non-routine problems for a quarter or semester course. Each problem...
typically requiring at least one week to complete and two-thirds of the problems being solved in co-operative groups. Students would be required to, orally and/or in writing, document the process for solving each problem. Approximately forty percent of the problems would involve content not typically considered mathematical, including problems that students individually define and solve. The curriculum should include an introductory unit that describes the three steps of a non-routine problem, gives several examples of the steps applied in a variety of fields, and has the students research additional examples. Field-testing indicates that the following guidelines are important when implementing such a curriculum of non-routine problems:

1. The most effective way to improve students' ability to solve non-routine problems is to repeatedly put them in a situation in which they work on a non-routine problem at an appropriate level of difficulty, and then discuss or process the problem as a class.

2. In an effective sequence, it is normal that each problem is difficult; that the student would not be clear about how to solve the problem initially, at times feel as if the problem is not solvable, and would need to persist until the problem becomes clear.

3. The teacher needs to provide a structure that allows students to work on problems at an appropriate level of difficulty. We can manipulate the level of support we provide students when solving the problem. In previous work [10], I define seven levels of support that form an instructional sequence. For example, at the fourth level, students are given a non-routine problem to solve, and groups brainstorm ideas for each of the three steps for teacher review before attempting each step.

4. A co-operative group model is consistent with the purposes of this curriculum. Students benefit from being exposed to the thinking of other students, and co-operative group work provides a supportive atmosphere for dealing with the natural difficulty of the problems [6].

5. The curriculum needs to emphasize problems from a variety of fields and problems that are relevant to the student's life, otherwise students are unlikely to transfer the problem-solving skills to their life.

6. For most students, non-routine problems that do not require significant content prerequisite skills are more effective initial problems than non-routine problems that require significant content prerequisite skills [9]. Appropriate instruction in solving non-routine problem does not have to be delayed because of lack of content mastery [1].

To make these guidelines clearer, I will share an example of a sequence of non-routine problems. In teaching an undergraduate mathematics course, the problems included:

1. solving one of three practical problems: planning a trip to a nearby resort, buying a car, or selecting a restaurant for a special occasion,
2. calculating the area of an irregular shape on graph paper,
3. determining the distribution of four different color marbles based on random samples,
4. determining grades based on raw test scores and some minimal information,
5. identifying significant improvements in one's home consistent with an ecological perspective,
6. constructing three solid figures without a "lid" that maximize volume, using one piece of construction paper for each solid,
7. Five Calculations, for each of 20 numbers, between 100 and 900, provided by the teacher, reduce each number to 0 with five or less calculations involving addition, subtraction, multiplication, and/or division with the numbers 1 to 9 only; Students are marked on points (based on the number of steps) and speed,
8. constructing the most pleasing (aesthetically) rectangle, with one dimension 6", from a piece of construction paper – the winning rectangle is usually close to the Golden Mean,
9. graphing nonlinear equations without a graphing calculator (e.g., plotting points and looking for patterns),
10. two individual problems that are meaningful in the student's life.

To clarify the individual problems, I will cite a few examples of undergraduate student selected problems: planning a vacation (e.g., planning a cruise for less than $1000); selecting the best engagement ring; finding a new job; relieving my frequent headaches; losing weight effectively; and developing a profitable tutoring service. Perhaps the most interesting problem was to find a humane way to reduce the number of squirrels on the student's property – I will not try to explain the somewhat complicated, effective solution.

**Conclusion**

Hopefully, I have conveyed my enthusiasm for this type of approach to teaching the process of problem-solving. I offer one last suggestion in implementing such a curriculum -- allow yourself to be open to the richness of the problems, to the variety of solutions, and to the opportunities for growth that the student efforts will create. If you have any questions or comments, please contact me at rlondon@csusb.edu.

**References**


